Geodesics of Kerr spacetime

Rotating black holes are described by the Kerr metric, a stationary (time independent, but not time-reversal-invariant as in the static case), axially symmetric solution to the Einstein equations. In Boyer-Lindquist coordinates, the Kerr spacetime is given by the line element

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4aMr\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(\Delta + \frac{2Mr(r^2+a^2)}{\Sigma}\right) \sin^2\theta d\phi^2$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$, and where and the black hole is parametrized by two quantities: its mass, M, and its angular momentum per unit mass, a. In the limit $a \to 0$ the Kerr metric reduces to the Schwarzschild solution. In this exercise, we will study geodesics of Kerr spacetime. We will make our life easy by using Killing vectors and tensors.

1. A *Killing vector field* is one for which the Lie derivative of the metric with respect to a vector K is 0, meaning

$$\mathcal{L}_K g_{ab} \equiv K^c \partial_c g_{ab} + g_{ac}\partial_b K^c + g_{bc}\partial_a K^c = 0.$$

(This means that in "adapted coordinates" the metric is independent of the coordinate along the integral curves of K.)

(a) Show that for our (torsion-free) Christoffel connection this equation may be written as

$$\nabla_{(a} K_{b)} = 0.$$

(b) Suppose that the smooth covector field K_a satisfies Killing's equation: $\nabla_{(a}K_{b)} = 0$. Show that

$$\nabla_c \nabla_d K_a = R^b{}_{cda} K_b.$$

Deduce that K_b satisfies the following modified vector wave equation:

$$\nabla^c \nabla_c K_b + R^c{}_b K_c = 0.$$

(c) Show that if K_b is a Killing vector and U^a is the tangent to an affinely parametrised geodesic then $U^a K_a$ is a constant along the geodesic.

(d) Show that $\xi^a_{(t)} = [1, 0, 0, 0]$ and $\xi^a_{(\phi)} = [0, 0, 0, 1]$ are Killing vectors of Kerr spacetime.

(e) Using part (c), we therefore have two conserved quantities, the energy $\mathcal{E} = -u_t$ and angular momentum $\mathcal{L} = u_\phi$. Write down the equations for $\frac{dt}{d\tau}$ and $\frac{d\phi}{d\tau}$ in terms of these constants.

2. A Killing *tensor* is a symmetric tensor $K_{ab} = K_{(ab)}$ that satisfies

$$\nabla_{(a}K_{bc)} = 0.$$

The Kerr spacetime admits a Killing tensor given by

$$K^{ab} = 2\Sigma l^{(a}n^{b)} + r^2 g^{ab}$$

where we have introduced the null tetrad

$$l^a = \left[\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right], \quad n^a = \frac{1}{2\Sigma}\left[r^2+a^2, -\Delta, 0, a\right],$$

$$m^a = \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left[ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right], \quad \bar{m}^\alpha = \frac{1}{\sqrt{2}(r-ia\cos\theta)}\left[-ia\sin\theta, 0, 1, -\frac{i}{\sin\theta}\right].$$

(a) Show that K_{ab} is a Killing tensor of Kerr spacetime.

(b) Show that if K_{ab} is a Killing tensor and u^a is the tangent to an affinely parametrized geodesic, then $K_{ab}u^a u^b$ is a constant along the geodesic.

(c) Show that the two Killing vectors of Kerr spacetime are related to the Killing tensor by

$$K^a{}_b\xi^b_{(t)} = -\left[a^2\xi^b_{(t)} + a\xi^a_{(\phi)}\right].$$

(d) Show that the Kerr Killing tensor may alternatively be written as

$$K^{ab} = 2\Sigma m^{(a}\bar{m}^{b)} - a^2\cos^2\theta g^{ab}.$$

(e) Hence, or otherwise, show that a constant of motion for timelike geodesics of Kerr spacetime is given by

$$K = u_\theta^2 + a^2\cos^2\theta + a^2\mathcal{E}^2\sin^2\theta + \frac{\mathcal{L}^2}{\sin^2\theta} - 2a\mathcal{E}\mathcal{L}.$$

(f) Using the timelike condition $g_{\alpha\beta}u^\alpha u^\beta = -1$, obtain an equation for u_r in terms of the three constants of motion.

(g) Show that the existence of the constant K is consistent with there being geodesics of Kerr spacetime that are purely in the equatorial plane and determine its value in that case.